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An Outline of Possible In-course Diagnostics for Derivatives and Integrals of Functions

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ABSTRACT The research reported in this paper was conducted in an attempt to provide support to first year mathematics students who took a core calculus module for which the relatively low pass rate was of concern. The objective was to make available suitable diagnostic questions to students so that they could detect their strengths and/ or weaknesses in particular sections before they took the formal assessments for those sections. In this paper, the researcher documented his findings on the formulation of expected student learning outcomes for derivatives and integrals, in the context of calculus. Those were arrived at after conducting document analyses of the aim, content and past assessment papers for the module. Then the literature review and the expected student learning outcomes were used to document sample diagnostic questions for the sections on derivatives and integrals that should facilitate student learning of concepts in those sections.

INTRODUCTION

At the University of KwaZulu-Natal the relatively low pass rates of the first year Introduction to Calculus module (Math130) was of concern. Previous papers by this researcher looked at the pre-course diagnostics for differential calculus (Maharaj and Wagh 2014), in-course diagnostics for functions (Maharaj and Wagh 2015a) and elementary logic, and limit and continuity of a function (Maharaj 2015). In this paper the focus is to outline an in-course diagnostics of the section on derivatives and integrals of functions, the latter for basic functions. This aims at assessing the strengths and shortcomings of technical knowledge and skills of the student. The diagnostic tests would not be for grading students but rather to provide feedback on their strengths and weaknesses with regard to content and skills relevant to derivatives and integrals of functions. The researcher provides detailed learning outcomes for the topic on derivatives and some integrals for the first year differential calculus module, as offered at the University of KwaZulu-Natal (UKZN).

The research questions for this paper were: What are the expected learning outcomes with regard to the sections on derivatives and integrals of functions? How could in-course diagnostics on derivatives and integrals of functions be developed?

Review of Literature

The focus here was on diagnostic testing, derivatives (including the chain rule) and integral of a function.

Diagnostic Testing

The papers, "An outline of possible pre-course diagnostics for differential calculus" (Maharaj and Wagh 2014) and "An outline of possible in-course diagnostics for functions" (Maharaj and Wagh 2015) discuss in detail the rationale for diagnostic testing and how one should go about formulating diagnostic questions. For the convenience of the reader the main points are summarized as follows.

- Clear learning outcomes for sections should be documented and these should be made public (Council of Regional Accrediting Commissions 2004).
- The formulation of the diagnostic questions should be guided by the relevant expected learning outcomes (Adam 2006).
- Pre-course paper-based or computer-based diagnostic testing has been and continues to be used by many institutions in the United Kingdom (Learning and Technology Support Network Maths Team Project 2003).
- There is evidence that diagnostic testing has led to improvement of student performance (Betts et al. 2011).

- Diagnostic tests could help a student if the feedback is given relating to the strengths and weaknesses of the student, which could help him or her plan and take remedial measures to attend to identified weaknesses (The California State University 2012).

The literature review indicated that many institutions used diagnostic testing to gauge the readiness of students to study calculus. In this paper the focus was on the formulation of expected learning outcomes and diagnostic questions for in-course diagnostics relating to derivatives and anti-derivatives of functions. In the opinion of this researcher these could improve the performance of students studying those sections, in particular at the University of KwaZulu-Natal and generally in developing countries.

Derivatives of Functions, Including the Chain Rule

There are a number of studies on students' understanding of the concept of a derivative of a function. For details on some of those studies the reader could refer to Maharaj (2013). Some of the important points are summarized for the reader as follows.

1. The derivative is a difficult concept for many students to understand (Orton 1983; Uygur and Özdas 2005).
2. A commonly used description of the derivative is the following: gradient of a function $f(x)$ *at* x_0 is the slope of the tangent line to the curve f at the point $(x_0, f(x_0))$. Further, one should be careful when distinguishing between a description of a concept (which specifies some properties of that concept) and the formal concept definition (Giraldo et al. 2003).
3. The students' understanding of the derivative can be improved if they are exposed to several different kinds of representations, to process the derivative (Hähkiöniemi et al. 2004). For example, this could be done by exposing students to different perceptual (for example, rate of change of the function from the graph, steepness of tangent) and symbolic (for example, differentiation rule, slope of a tangent) representations of the concept of a derivative.
4. Roorda et al. (2009) found that growth in understanding depends on a variety of connections, both between and within representations, and also between a physical application and mathematical representations. So this implies that there should be a focus on representations and their relevant connections, as part of understanding derivatives.
5. It seems that students prefer the graphical representation in tasks and explanations about derivatives (Zandieh 2000). This was also noted by Tall (2010) who made a strong argument for direct links between visualization and symbolization when teaching the concept of a derivative.
6. If the function considered is a composite function, then students' difficulties with the derivative increase and get worse (Tall 1993).

This could be the reason for the chain rule being one of the hardest ideas to convey to students in calculus (Gordon 2005; Uygur and Özdas 2007). The difficulties with the chain rule and its applications, for a large number of students, could be attributed to their difficulties in dealing with composition and decomposition of functions (Clark et al. 1997). So, the understanding of composition of functions is an integral part to understanding the chain rule (Webster 1978; Cottrill 1999; Horvoth 2007).

Anti-derivative, Integral of a Function

A number of studies (for example, Orton 1983; Ubuz 1993; Abdul-Rahman 2005; Haciomeroglu et al. 2009; Sevimli and Delice 2010; Maharaj 2014) focused on student understanding of integration. The following conclusions could be made from those studies.

a. Differentiation can be viewed as a *forward* process and the difficulties faced by students in this concept are not as complicated as those in the reverse or backward process of integration (Abdul-Rahman 2005). This implies that teaching needs to relate the anti-derivative concept with that of the derivative. For example, $\int f(x)dx$ represents the general anti-derivative of $f(x)$ so $\int f(x)dx=F(x)+C$ provided $F'(x)=f(x)$.

b. Integration has a dual nature. It is both the inverse process of differentiation and a tool for calculation, for example, area. For example, with regard to this inverse process the derivative of a^{kx} is $k \ln a . a^{kx}$, therefore an anti-derivative of a^{kx} is $[1/(k \ln a)]\, a^{kx}$ so $\int a^{kx}\, dx = [1/(k \ln a)]\, a^{kx} + C$.

c. Orton (1983) suggested that student errors with regard to the concept of integration could arise from their failure to appreciate the relationships involved in the problem or to their grasping of a principle essential to the solution, or failure to take account of the constraints laid down in what was given, or failure to carry out manipulations (though the principle involved may have been understood). Further, some errors could involve elements of more than one of these.

d. Kiat (2005) in a study, found students had difficulty with questions on integration of trigonometric functions and also applying integration to evaluate plane areas. The students generally lacked both conceptual and procedural understanding of integration. The errors committed were technical errors, primarily attributed to the students' lack of specific mathematical content knowledge.

e. Visualization in the graphical context could help students understand the relations between differentiation and integration (Ubuz 1993).

f. A student's use of area under a curve is helpful in problem solving only when a deeper understanding of the structure behind the definite integral is present (Sealey 2006).

g. There should be a focus on the relationship between the graphical and symbolic integral representations. Sevimli and Delice (2010) argued that this could increase the performance of solving definite integral problems.

h. The students' understanding can be enriched by changing thinking processes and establishing reversible relations between graphs of functions and their derivative or definite integral graphs (Haciomeroglu et al. 2009).

The implication of the above is that a variety of representations should be used and students should be encouraged to engage with a flexibility of mathematical conceptions (Andresen 2007; Maharaj 2010) of the derivative of the function f and the its integral $\int f(x)dx$.

Conceptual Framework

This study was guided by the literature review and the following principles:

a. There is a conceptual hierarchy in the body of mathematics. This principle informed the formulation of the student expected learning outcomes and the development of sample diagnostic questions.

b. It is important for the learning outcomes for the unit or module to be clearly documented. Further, students should know explicitly at the outset the learning outcomes expected of them. In this study these were the reasons for clearly documenting the learning outcomes that were formulated. Those learning outcomes could then be easily made available to students in an electronic format.

c. For effective learning to occur it is not good enough for an instructor (teacher, lecturer or tutor) to be aware of the technical knowledge outcomes of a course or module. The documented learning outcomes should guide the formulation of suitable diagnostic questions for students. This is the procedure that was followed during the research that is reported on in this paper. The documented student learning outcomes and sample diagnostic questions arrived at are reported on in the findings and discussion section of this paper.

d. When students attempt the diagnostic questions there should be provisions for remedial activity, to overcome their identified shortcomings. The reason for developing the diagnostic questions was to provide students with a means of identifying their strengths and weaknesses for the sections on derivatives and integrals of functions. Students could then take the necessary remedial measures before taking formal assessments based on the relevant sections for derivatives and integrals of functions.

METHODOLOGY

The literature review, conceptual framework and study of the aims and content for the differential calculus (Math130) module informed the methodology. The researcher looked at the aim and content as indicated in the handbook of the Faculty of Science and Agriculture (2010), which is in the public domain. These are indicated below, and the parts in italics are the researcher's emphasis.

Aim: To introduce and develop the Differential Calculus as well as the fundamentals of proof technique and rudimentary logic.

Content: Fundamental Concepts like elementary logic, proof techniques. Differential Calculus functions, graphs and inverse functions, limits and continuity, *the derivative*, *techniques of differentiation*, *applications of derivatives*, *anti-derivatives*.

The researcher then used his experience relating to teaching at both secondary and tertiary education institutions to formulate and document the following.

a. In-course expected learning outcomes for the sections on *the derivative of a function*, *techniques of differentiation*, *applications of derivatives*, *anti-derivatives (integrals) of some standard functions*. Such outcomes for the Math130 module were formulated by studying its aim and content (as indicated above), and also past assessment questions.
b. In-course diagnostics for *the derivative*, *techniques of differentiation*, *applications of derivatives*, *anti-derivatives*. The researcher used the learning outcomes identified to formulate questions on course content for the derivative of a function, techniques of differentiation, applications of derivatives, and anti-derivatives or integrals of standard functions.

FINDINGS AND DISCUSSION

These are presented in the following order, the derivative of a function, and anti-derivatives and integrals. For each of these, the learning outcomes are stated and then the sample diagnostic questions that were formulated are given.

The Derivative of a Function

First, the expected learning outcomes that were formulated are presented, followed by the sample diagnostic questions. The sample diagnostic questions are indicated under the following subheadings: derivatives from first principles, techniques for differentiation, and applications of derivatives.

Learning Outcomes

The researcher expects a student to be able to:

- Recall the defining condition for the existence of the derivative of a function at a given point or a given domain
- Recall the defining condition for the existence of the derivative of a function on the whole domain
- Determine if a given function has a derivative at a given point in its domain or on the entire domain
- Compute the derivatives of standard functions from first principles
- Deduce and recall the laws of derivatives
- Recall the derivatives of standard functions
- Compute the derivatives of new functions
- Recognize standard observations emerging from the derivative of a function. This includes comparison of the graphs of a function and that of its derivative with the focus on derivative is constant on an interval, derivative is increasing or decreasing on an interval, derivative is zero at a point, derivative changes sign, derivative is a multiple of the reciprocal of the variable, derivative is an exponential, and derivative is periodic.

Diagnostic Questions

This concerns defining conditions for a derivative of a function to exist at points in its domain and on the whole domain, laws of derivatives, derivatives of standard and new functions, observations emerging from the derivative of a function and their applications to standard and new functions.

Sample Questions on Derivatives from First Principles

Past experience indicated that students confuse situations where the calculation of the derivative is required from first principles. Informed by the relevant learning outcomes that were documented, the diagnostic questions arrived at are indicated in Table 1. In framing those questions the researcher drew from the literature review on the importance of the formal concept definition (Giraldo et al. 2003).

Sample Questions on Techniques of Differentiation

Using the relevant expected learning outcomes, the researcher arrived at diagnostic questions on the rules for differentiation of new functions, including exponential and logarithmic functions. A sample of those questions is given in Table 2. Since the chain rule is one of the hard-

Table 1: Derivatives from first principles

No.	Question	Answer
1.	State the defining condition for the derivative of a function *f(x)* to exist at $x = a$ in its domain.	Condition: $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.
2.	State the defining condition for the derivative of a function to exist on its whole domain.	Condition: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists for each value of in its domain.
3.	For the function $g(x) = \sqrt{x}$ determine from first principles the derivative, *g(x)*, and state the values for which it exists.	$g'(x) = \frac{1}{2\sqrt{x}}$ Domain of *g(x)* is [0, ∞). *g'(x)* exists for all *x* in (0, ∞) .

est ideas to convey to students in calculus (Gordon 2005; Uygur and Özdas 2007), the researcher formulated a number of questions based on the application of this rule, see question 6. Those questions require the application of the chain rule in the context of different mathematical structural representations. The researcher felt that student growth in understanding the techniques for differentiation could be promoted by exposing them to the application of the chain rule in the context of various mathematical representations (Roorda et al. 2009). A study of those questions indicate that the application of the chain rule is embedded in mathematical structures requiring, for example, first applying the power rule or the rule quotient rule for differentiation [see questions 6(a) and 6(b) respectively].

Table 2: Differentiation of new functions, including exponential and logarithmic

A number of questions based on differentiation of trigonometric and other functions (see Table 3) were formulated. Since differentiation and integration are reverse processes, the researcher noted the findings of Kiat (2005) who, in a study, found students had difficulty with questions on integration of trigonometric functions. It was felt that one needs to first focus on the differentiation of trigonometric functions and later make the relevant links with integration. Further, note that once again the focus was on many different mathematical structures, which required the application of the chain rule (see questions 8 and 9). The researcher felt that it was important for students to analyze the given mathematical structure, first detect from the overall structure and then the embedded structure the rule or rules for differentiation to be applied (see question 8). Once this type of mathematical cognition is attained then they could concentrate on the applications of the relevant techniques for differentiation (see question 9).

The questions that were formulated for applications of the derivative are indicated in Table 4. Those question types were arrived at from the expected learning outcomes that were documented. The researcher was also informed by the important points that were noted in the literature review with regard to the students' understanding of the derivative concept. Some of those points were as follows.

a. The students' understanding of the derivative can be improved if they are exposed to several different kinds of representations (Hähkiöniemi 2004). See questions 11, 12 and 13.
b. Roorda et al. (2009) found that growth in understanding depends on a variety of connections, both between and within representations. The questions in Table 4 were designed to provide a variety of connections between and within representations.
c. It seems that students prefer the graphical representation in tasks and explanations about derivatives (Zandieh 2000; Tall 2010). That was one of the reasons for giving the graphical representation and the defining equation of the rational function in question 11. Another was the argument by Mahir (2010) that learning about functions and their graphical interpretation with suitable connections could

Table 2: Differentiation of new functions, including exponential and forgarithmic

4.	If $f(x)$ and $g(x)$ are functions defined on their domains, complete the follows laws for finding derivatives of functions:	
	a) For some constant k, $y = k \Rightarrow y' =$ __.	a) 0
	b) For some constant $k, y = kf(x) \Rightarrow y' =$ ___.	b) $kf'(x)$
	c) $y = f(x) \pm g(x) \Rightarrow y' =$ ____.	c) $f'(x) \pm g'(x)$
	d) $y = f(x).\, g(x) \Rightarrow y' =$ _______.	d) $f'(x).\, g(x) + f(x).\, g'(x)$ $\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$
	e) $y = \frac{f(x)}{g(x)} \Rightarrow y' =$ __________	e) $f'(g(x)).\, g'(x)$
	f) $y = f(g(x)) \Rightarrow \frac{dy}{dx} =$ ______	
5.	Complete by stating the derivative of the following standard functions:	
	a) $f(x) = x^n$ for $n \neq 0$	a) $f'(x) = nx^{n-1}$, $n \neq 0$
	b) For $n \neq 0, y = [g(x)]^n$	b) $y' = n[g(x)]^{n-1}g'(x)$
	c) $h(x) = e^{kx}$, k a non-zero constant.	c) $h'(x) = ke^{kx}$
	d) $y = a^{kx}$, k a non-zero constant and $a > 0$ $but\ a \neq 1$.	d) $h'(x) = k \ln a .\, a^{kx}$
	e) $y = \ln(kx)$, k a non – zero constant	e) $y' = \frac{1}{x}$
	f) $y = \log_a(kx)$, k a non-zero constant and $a > 0\ but\ a \neq 1$.	f) $y' = \frac{1}{(\ln a)x}$
6.	Find the derivatives of the following functions:	
	a) $y = 4x^2 - x + 3$	a) $y' = 8x - 1$
	b) $f(x) = \sqrt{3x^4 + 1}$	b) $f'(x) = \frac{6x^3}{\sqrt{3x^4+1}}$
	c) $h(x) = \frac{6x-2}{3x^2-2x}$	c) $h'(x) = \frac{-18x^2+12x-4}{(3x^2-2x)^2}$
	d) $g(x) = 3e^{4x} + (5x-1)^e$	d) $g'(x) = 12e^{4x} + 5e(5x-1)^{e-1}$ $y' = \frac{10x+1}{5x^2+x}$
	e) $y = \ln(5x^2 + x)$	e) $y' = -\ln 3 .\, (3)^{-x} - \frac{1}{\ln 10(1-x)}$
	f) $y = (3)^{-x} + \log(1-x)$	f) $y' = -300(x^2-x)^{\frac{e}{2}}(13x)^{99} + \frac{e}{2}(2x-1)(x^2-x)^{\frac{e}{2}-1}(1-3x)^{100}$
	g) $y = (x^2-x)^{\frac{e}{2}}(1-3x)^{100}$	

ensure that students effectively learn concepts in calculus.

The researcher formulated question 12 to facilitate the development of such connections in the context of calculus concepts.

Anti-derivatives and Integrals

The expected learning outcomes that were formulated followed by the sample diagnostic questions are now presented. Note that the sample diagnostic questions are indicated under the following subheadings: anti-derivatives, indefinite integrals and definite integrals.

Learning Outcomes

The researcher believes a student should be able to:

- Recall the defining conditions of an anti-derivative of a function and be able to recall the symbol for anti-derivative
- Recall the anti-derivative of standard functions
- Recall laws of anti-derivatives
- Compute anti-derivatives of new functions if the anti-derivative is expressible in terms of standard functions
- Recall the standard observations emerging from the anti-derivative: the derivative of the anti-derivative gives the integrand, any two anti-derivatives of a function differ by a constant, the area between the graph of a function on an interval and the x-axis is the difference of the values of the anti-derivative of the function at the endpoints of the interval.

Table 3: Differentiation of trigonometric and other functions

7.	For each of the following trigonometric function find the derivative $\frac{dy}{dx}$ if k is a non-zero constant.	a)	$k\cos(kx)$
	a) $y = \sin(kx)$	b)	$-k\sin(kx)$
	b) $y = \cos(kx)$	c)	$k(\sec(kx))^2$
	c) $y = \tan(kx)$	d)	$-k\csc(kx)\cot(kx)$
	d) $y = \csc(kx)$	e)	$k\sec(kx)\tan(kx)$
	e) $y = \sec(kx)$	f)	$-k(\csc(kx))^2$
	f) $y = \cot(kx)$		
8.	Which rules will you use to compute the derivatives of the following functions with respect to x:	a)	Chain rule
	a) $\cos^3 x$	b)	Addition rule, product rule and chain rule
	b) $3x^7\tan(2x) + 4$	c)	Chain rule
	c) $\sqrt[5]{\sec(\sin x)}$	d)	Laws of indices and chain rule
	d) $\frac{1}{\sqrt{x^2 - x}}$	e)	Quotient rule and chain rule
	e) $\frac{(x^5 - x + 1)^{234}}{(x^6 + x - 2)^{567}}$	f)	Implicit differentiation
	f) $\frac{x^2}{4} + y^2 = 1$		
9.	Find the derivatives of the function in question 8 above.	1.	$-3\sin(x)(\cos(x))^2$
		2.	$6x^7(\sec(2x))^2 + 21x^6\tan(2x)$
		3.	$\frac{1}{5}(\sec(\sin x))^{-\frac{4}{5}}\sec(\sin x)\tan(\sin x)\cos x$ or $\frac{\cos(x)\sin(\sin(x))}{5(\cos(\sin(x)))^{\frac{6}{5}}}$
		4.	$-\frac{2x-1}{2(x^2-x)^{\frac{3}{2}}}$
		5.	$\frac{234(x^5 - x + 1)^{233}(5x^4 - 1)(x^6 + x - 2)^{567} - 567(x^6 + }{(x^6 + x - 2)^{1134}}$
		6.	$y' = -\frac{x}{4y}$

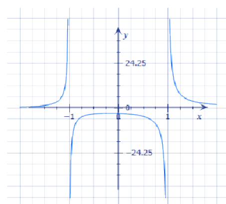
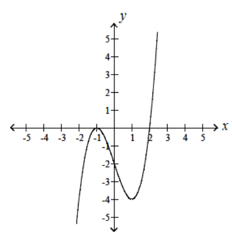
Diagnostic Questions

This concerns the defining conditions for an anti-derivative of a function, laws for integrals, integrals of standard and new functions, and the area interpretation of a definite integral. Based on the literature review, conceptual framework and the learning outcomes arrived at the sample diagnostic questions that were formulated are indicated in Tables 5, 6 and 7. For example, the literature review on the suggestions of Abdul-Rahman (2005) implied that teaching needs to relate the anti-derivative concept with that of the derivative. See Table 5 (Questions 1 and 2).

The formulations of the diagnostic questions in Table 6 were guided by the suggestions of Orton (1983) on the type of student errors with regard to the concept of integration. The researcher also noted the work of Kiat (2005) who found students had difficulty with questions on integration of trigonometric functions (see Question 5) and applying integration to evaluate plane areas (see Table 7, Questions 7 and 10).

The diagnostic questions for definite integrals indicated in Table 7 were arrived at by focusing on the following from the literature review. A student's use of area under a curve is helpful in problem solving only when a *deeper*

Table 4: Applications of derivatives

<table>
<tr><td>10.</td><td>What is meant by point of inflection of a curve?</td><td>A point on the curve where the concavity of the curve changes.</td></tr>
<tr><td>11.</td><td>Study the graphical illustration for a portion of the graph defined by the equation $y = \frac{x+3}{x^2-1}$. For this function determine the y-intercept, the defining equations of the asymptotes, the intervals over which the graph is increasing or decreasing, the intervals of concavity, and the local extrema.

</td><td>y-intercept is -3; vertical asymptotes are defined by $x = -1,\ or\ x = 1$, horizontal asymptote x-axis defined by $y = 0$; increasing over $(-\infty, -1) \cup (-1,0)$, decreasing over $(0,1) \cup (1, \infty)$; concave up over $(-\infty, -1) \cup (1, \infty)$, concave down over $(-1,1)$; local maximum of -3 at $x = 0$.</td></tr>
<tr><td>12.</td><td>The sketch shows the graph of the function $y = h'(x)$. Interpret the sketch and use it to answer the following questions.


a) Identify the extrema of the function h.
b) Identify the values of x where the points of inflection of curve of h occur.
c) Discuss the concavity of the graph of h.</td><td>a) A relative minimum occurs at $x = 2$.
b) -1; 1
c) concave up on $(-\infty, -1) \cup (1, \infty)$ and concave down on $(-1, 1)$</td></tr>
<tr><td>13.</td><td>The function $M(x) = -\frac{1}{45}x^2 + 2x - 20;\ 30 \le x \le 65$; isan approximation for the number of kilometres per litre of fuel used by a new prototype car, when driven at a speed of x kilometers per hour.
a) Calculate the speed intervals over which the</td><td>a) increases on the speed interval $(30, 45)$ and decreases on the speed interval $(45, 65)$
b) Absolute maximum is 25 km per litre. Achieved at a speed of 45 km per hour.</td></tr>
</table>

understanding of the structure behind the definite integral is present (Sealey 2006). See Questions 7, 8 and 10. There should be a focus on the relationship between the graphical and symbolic integral representation, see Questions 7 and 10, since these could increase student performance of solving definite integral problems as was argued by Sevimli and Delice (2010).

Creative Thinking

The learning outcomes and sample diagnostic questions that were arrived at are now presented. In the opinion of the researcher this aspect is often neglected in the teaching and learning situation.

Table 5: Concept of the anti-derivative

a)	What is the defining condition for a function to be an anti-derivative of a function $f(x)$?	$h'(x) = f(x)$
b)	Interpret $\int f(x)dx$	The general anti-derivative (or integral) of the function $f(x)$ with respect to the variable x.
c)	Complete: Two anti-derivatives of a function differ by a ______.	constant

Table 6: Integration rules, applications

4.	For functions $f(x)$ and $g(x)$ complete the following:			
	a)	$\int kf(x)\,dx =$	a)	$k\int f(x)dx$
	b)	$\int[f(x) \pm g(x)]\,dx =$	b)	$\int f(x)\,dx \pm \int g(x)\,dx$
5.	If a and k are non-zero constant, complete the following integrals of standard functions:			
	a)	$\int kdx =$	1.	$kx + C$
	b)	For $n \neq -1$, $\int x^n\,dx =$	2.	$\frac{1}{n+1}x^{n+1} + C$
	c)	$\int x^{-1}dx =$	3.	$\ln x + C$
	d)	For $a > 0$ and $a \neq 1$, $\int e^{ax}\,dx =$	4.	$\frac{e^{ax}}{a} + C$
	e)	$\int a^{kx}\,dx =$	5.	$\frac{a^{kx}}{k \ln a} + C$
	f)	$\int \sin(ax)\,dx =$	6.	$-\frac{1}{a}\cos(ax) + C$
	g)	$\int \cos(ax)\,dx =$	7.	$\frac{1}{a}\sin(ax) + C$
	h)	$\int \sec^2(ax)\,dx =$	8.	$\frac{1}{a}\tan(ax) + C$
	i)	$\int \csc^2(ax)\,dx =$	9.	$-\frac{1}{a}\cot(ax) + C$
	j)	$\int \sec(ax)\tan(ax)\,dx =$	10.	$\frac{1}{a}\sec(ax) + C$
	k)	$\int \csc(ax)\cot(ax)\,dx =$	11.	$-\frac{1}{a}\csc(ax) + C$
6.	Compute the following integrals:		1.	$2x + \frac{x^4}{4} - 3\ln x + C$
	a)	$\int \left(2 + x^3 - \frac{3}{x}\right)dx$	2.	$\frac{4e^{3t}}{3} - \frac{7^{-2t}}{2 \ln 7} - \frac{2}{3}t^{\frac{3}{2}} + C$
	b)	$\int\left(4e^{3t} + 7^{-2t} - \sqrt{t}\right)dt$	3.	$ex - \frac{1}{e}\cos(ex) + C$
	c)	$\int[e + \sin(ex)]dx =$	4.	$\int \sec x \tan x\,dx = \sec x + C$
	d)	$\int \frac{\sin x}{1-\sin^2 x}dx =$		

Learning Outcomes

The researcher expects students to be able to frame non-routine questions, and identify applications of mathematical concepts studied to various contextual situations in their surroundings. Each of these could contribute to the development of critical thinking and hence understanding in calculus (Maharaj and Wagh 2015b). Understanding in the context that the

Table 7: Definite integrals, rules and area interpretation

7.	Use the area interpretation of the definite integral to evaluate the following integrals: a) $\int_{-2}^{0}\sqrt{4-x^2}dx$ b) $\int_{-1}^{1}\lvert x\rvert dx$ c) $\int_{0}^{2\pi}\sin\theta d\theta$	g) π h) 1 i) 0
8.	State the conditions for $\int_a^b f(x)\,dx$ to exist.	The function $f(x)$ must be continuous on the closed $[a,b]$.
9.	State all the properties specific to definite integrals.	For continuous functions $f(x)$ and $g(x)$ on a closed interval $[a,b]$, and k a non-zero constant the following hold: 1. $\int_a^a f(x)dx = 0$ 2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ 3. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ 4. For any c in $[a,b]$, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ 5. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$ 6. For an even function continuous on $[-b,b]$, $\int_{-b}^{b} f(x)dx = 2\int_0^b f(x)dx$ 7. For an odd function continuous on $[-b,b]$, $\int_{-b}^{b} f(x)dx = 0$.
10	Evaluate the following definite integrals: 1. $\int_0^{\frac{\pi}{2}}\cos x\,dx$ 2. $\int_{-\pi}^{\pi}\sin x\,dx$	a) 1 b) 0

Table 8: Questions to promote critical thinking

a)	Find situations to apply the knowledge of the derivative in the context of cooking.	Indicative answer: Deep frying fish or chips. Rate of heat transfer from flame to pot, pot to oil, oil to interior of fish or chips. This is the reason for specific recipes indicating cooking time and the temperature at which cooking should occur. Note if the rate of heat transfer to oil is too high as compared the rate of heat from the oil to the interior of the fish then this results in the exterior of the fish getting burnt while the interior remains raw.
b)	Find situations to apply the knowledge of the derivative in the context of trading.	Indicative answer: Rate of supply, rate of distribution, rate demand and rate of financing govern the profitability of trade.
c)	Find situations to apply the knowledge of the derivative in the context of your other fields of study.	
d)	Find situations to apply the knowledge of the derivative in the context of classroom learning.	

mental structure related to schema development will have to be further developed and refined to incorporate new connections that help the students to make sense of the topic (Arnon et al. 2014; Menary 2015).

Diagnostic Questions

This concerns framing of new questions about the surrounding that seeks to apply the content of the course. Sample diagnostic questions to promote critical thinking are indicated in Table 8.

CONCLUSION

The aim and the content of a core mathematics module offered to first year university students together with past formal assessment papers for that module were used to document expected student learning outcomes for derivatives and integrals of functions. Those learning outcomes were then used to develop sample diagnostic questions for the relevant concepts. The sample diagnostic questions that were arrived at for derivatives and integrals gave an insight into how the teaching and learning of these sections could be approached. In the opinion of this researcher, the documented material in this paper could benefit both students and instructors.

RECOMMENDATIONS

It is recommended that first year mathematics lecturers should use or modify the sample diagnostic questions arrived at for derivatives and integrals of functions. These could be given to students in the form of a hard copy or online quizzes. It is further recommended that such quizzes should be used in the context of assisting students to identify their strengths and weaknesses before students are formally assessed on the relevant sections. It is recommended that at UKZN the planning should focus on the implementation of these materials in an online format, so that further investigations on how the material arrived at impacts on the teaching and learning of students could be done. Others who feel the material could be beneficial are encouraged to do the same. The researcher will be interested in the experiences and finding of others.

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